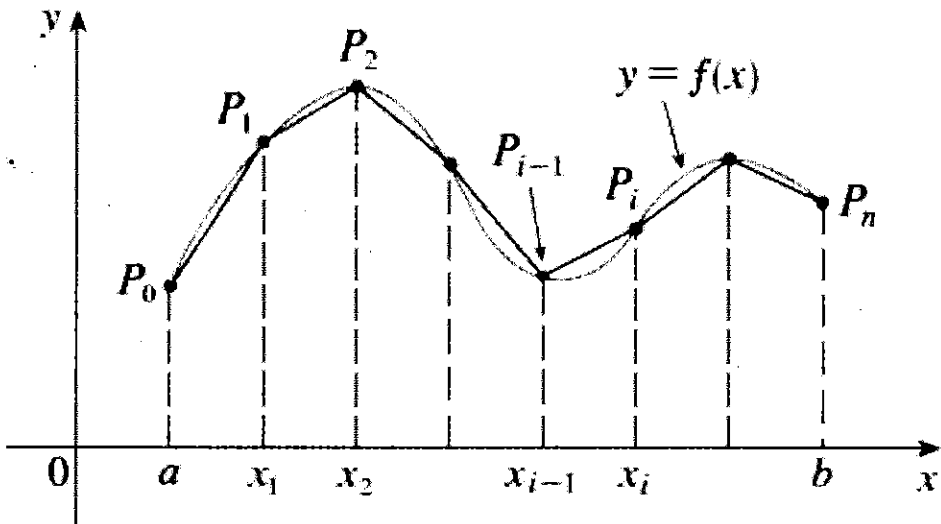


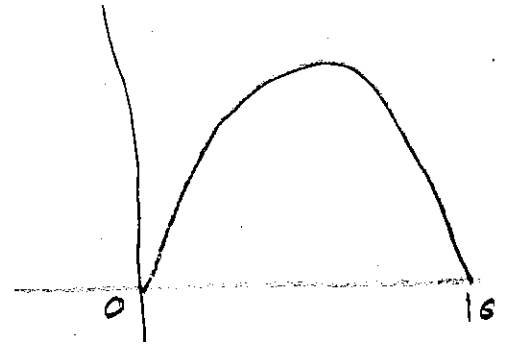
3.1 Arc Length

Goal: Given $y = f(x)$ from $x = a$ to $x = b$.
Want to find the **length** along the curve.

$$\text{Arc Length} = \int_a^b \sqrt{1 + (f'(x))^2} dx$$



Ex



$$y = -x^2 + 10x$$

How LONG IS THIS CURVE?

$$y' = -2x + 10$$

$$\begin{aligned} & \int_0^{10} \sqrt{1 + (-2x + 10)^2} dx \\ & \int_0^{10} \sqrt{1 + 4x^2 - 40x + 100} dx \\ & \int_0^{10} \sqrt{4 \left(x^2 - 10x + \frac{101}{4} \right)} dx \\ & 2 \int_0^{10} \sqrt{x^2 - 10x + 25 - 25 + \frac{101}{4}} dx \\ & 2 \int_0^{10} \sqrt{(x-5)^2 + \frac{1}{4}} dx \end{aligned}$$

$$x - 5 = \frac{1}{2} \tan \theta \quad \dots \approx \boxed{51.7485}$$

$$dx = \frac{1}{2} \sec^2 \theta d\theta$$

Derivation:

1. Break into n subdivision:

$$\Delta x = \frac{b-a}{n}, \quad x_i = a + i\Delta x$$

2. Compute $y_i = f(x_i)$.

3. Compute the straight line distance from (x_i, y_i) to (x_{i+1}, y_{i+1}) .

$$\sqrt{(x_{i+1} - x_i)^2 + (y_{i+1} - y_i)^2}$$

$$= \sqrt{(\Delta x)^2 + (\Delta y_i)^2}$$

$$= \sqrt{(\Delta x)^2 \left(1 + \frac{(\Delta y_i)^2}{(\Delta x)^2}\right)}$$

$$= \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x}\right)^2} \Delta x$$

4. Add these distances up.

$$\text{Arc Length} = \lim_{n \rightarrow \infty} \sum_{i=1}^n \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x}\right)^2} \Delta x$$

Note that

$$\lim_{\Delta x \rightarrow 0} \frac{\Delta y_i}{\Delta x} = \text{slope of tangent} = f'(x)$$

$$\text{Arc Length} = \lim_{n \rightarrow \infty} \sum_{i=1}^n \sqrt{1 + (f'(x))^2} \Delta x$$

$$\text{Arc Length} = \int_a^b \sqrt{1 + (f'(x))^2} dx$$

Good news:

Arc length is important. And we found an integral to compute arc length, yeah!

Bad news:

The arc length integral almost always is something that can't be done explicitly (we have to approximate, *Simpson's rule*), too!

In homework, you see the few, *unusual* cases where you actually can compute arc length explicitly.

Here are most of the 8.1 HW questions

Find the arc length of

1. $y = 4x - 5$ for $-3 \leq x \leq 2$.

2. $y = \sqrt{2 - x^2}$ for $0 \leq x \leq 1$.

3. $y = \frac{x^4}{8} + \frac{1}{4x^2}$ for $1 \leq x \leq 2$.

4. $y = \frac{1}{3}\sqrt{x}(x - 3)$ for $4 \leq x \leq 16$.

5. $y = \ln(\cos(x))$ for $0 \leq x \leq \pi/3$.

6. $y = \ln(1 - x^2)$ for $0 \leq x \leq 1/7$.

Example:

$$y = 4x - 5 \text{ for } -3 \leq x \leq 2.$$

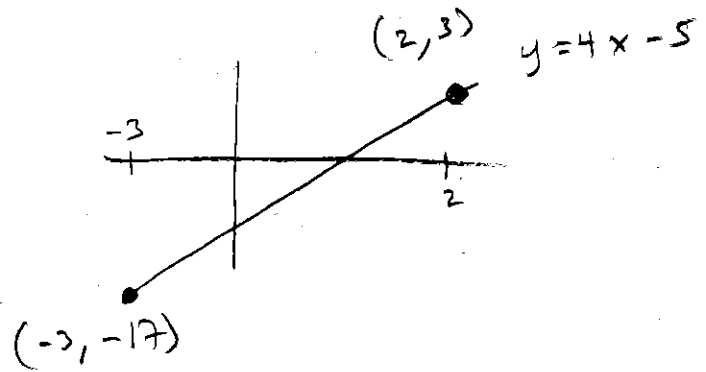
$$y' = 4$$

$$\int_{-3}^2 \sqrt{1 + (4)^2} dx$$

$$\sqrt{17} \times \Big|_{-3}^2$$

$$\sqrt{17} (2 - -3)$$

$$\boxed{5\sqrt{17}}$$



$$\sqrt{(2 - -3)^2 + (3 - -17)^2}$$
$$\sqrt{25 + 400}$$
$$\sqrt{425}$$

$$425 = 25 \cdot 17$$



Example:

$$y = \frac{x^4}{8} + \frac{1}{4x^2} \text{ for } 1 \leq x \leq 2$$

$$y' = \frac{1}{2}x^3 - \frac{1}{2}x^{-3}$$

$$\int_1^2 \sqrt{1 + \left(\frac{1}{2}x^3 - \frac{1}{2}x^{-3}\right)^2} dx$$

$$= \int_1^2 \sqrt{\left(\frac{1}{2}x^3 + \frac{1}{2}x^{-3}\right)^2} dx$$

$$= \int_1^2 \left(\frac{1}{2}x^3 + \frac{1}{2}x^{-3}\right) dx$$

$$= \left. \frac{1}{8}x^4 - \frac{1}{4}x^{-2} \right|_1^2$$

$$= \left(\frac{1}{8}(2)^4 - \frac{1}{4}(2)^{-2} \right) - \left(\frac{1}{8} - \frac{1}{4} \right)$$

$$= 2 - \frac{1}{16} + \frac{1}{8} = \frac{32-1+2}{16} = \frac{33}{16}$$

$$\frac{1}{8}x^4 + \frac{1}{4}x^{-2}$$

$$\left(\frac{1}{2}x^3 - \frac{1}{2}x^{-3}\right)\left(\frac{1}{2}x^3 - \frac{1}{2}x^{-3}\right)$$

$$1 + \left(\frac{1}{2}x^3 - \frac{1}{2}x^{-3}\right)^2$$

$$= 1 + \left(\frac{1}{4}x^6 - \frac{1}{2} + \frac{1}{4}x^{-6}\right)$$

$$= \frac{1}{4}x^6 + \frac{1}{2} + \frac{1}{4}x^{-6}$$

$$= \left(\frac{1}{2}x^3 + \frac{1}{2}x^{-3}\right)^2$$

Example:

$$y = \ln(\cos(x)) \text{ for } 0 \leq x \leq \pi/3.$$

$$y' = \frac{-1}{\cos(x)} (-\sin(x)) = -\tan(x)$$

$$\int_0^{\pi/3} \sqrt{1 + (-\tan(x))^2} dx$$

$$\int_0^{\pi/3} \sqrt{1 + \tan^2(x)} dx$$

$$\int_0^{\pi/3} \sqrt{\sec^2(x)} dx$$

$$\int_0^{\pi/3} \sec(x) dx$$

$$= \ln|\sec(x) + \tan(x)| \Big|_0^{\pi/3}$$

$$= \ln|\sec(\pi/3) + \tan(\pi/3)| - \ln|\sec(0) + \tan(0)|$$

$$= \ln|2 + \sqrt{3}| - \ln|1 + 0|$$

$$= \boxed{\ln(2 + \sqrt{3})}$$

$$\sin(\pi/3) = \sqrt{3}/2$$

$$\cos(\pi/3) = 1/2$$

$$\sin(0) = 0$$

$$\cos(0) = 1$$

Aside (don't need all this for this course)

In applications, Arc Length is used in motion (parametric) problems, which you will see a lot in Math 126:

$$x = x(t), y = y(t)$$

In this case, the same derivation from the beginning of class yields:

$$\text{Arc Length} = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2} dt$$

This gives the **distance** the object has traveled on the curve.

Very often, in motion problems we need:

$$s(t) = \int_0^t \sqrt{(x'(u))^2 + (y'(u))^2} du$$

which gives the distance traveled from time 0 to time t . This is called the **Arc Length (Distance) Function**.

Simple Example:

Consider

$$x = 3t, y = 4t + 2$$

where t is in seconds.

- Find the arc length from 0 to 10 sec.
- Find the arc length function.
- What is the derivative of the arc length function?

(a) $x' = 3, y' = 4$

$$\begin{aligned} \int_0^{10} \sqrt{(3)^2 + (4)^2} dt &= \int_0^{10} 5 dt \\ &= 5t \Big|_0^{10} \\ &= 50 \end{aligned}$$

b) $\int_0^t \sqrt{(3)^2 + (4)^2} du$
 $5u \Big|_0^t = 5t$

$$\boxed{s(t) = 5t}$$

$$s(t) = \int_0^t 5 dt$$

$$= 5t = \text{distance}$$

$$\boxed{s'(t) = 5} \text{ ft/sec} \leftarrow \text{speed}$$

$$s'(t) = \sqrt{(x'(t))^2 + (y'(t))^2}$$